# Detecting Objects with 2D Projections, Discrete Robotic Motion, Counting with Sensors

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joint work with Bill Moran, Doug Cochran, Stephen Howard, Clark Haynes, and Dan Koditschek

# Outline

- Detecting whether objects are rigidly connected from two dimensional projections (joint with Doug Cochran, Stephen Howard, and Bill Moran)
- A language and context for legged robotic motion (joint with Clark Haynes and Dan Koditschek)
- On counting with sensors (joint with Doug Cochran, Stephen Howard, and Bill Moran)

# Rigidly attached objects, and statement of the problem

- This section is based on joint work with Doug Cochran, Stephen Howard, and Bill Moran.
- A set of objects is assumed to be "moving around a center of mass".
- The problem is to devise a practical test from $2$-dimensional photographic data to 'decide' whether the objects are rigidly attached.

# First naive version of the problem

- Suppose that some objects are moving through time in an orbit around a center of mass a 'sun'. Find a 'test' to determine whether these objects are rigidly connected.
- If the objects are rigidly connected, their relative positions at any fixed time $t$ are determined by an element in the rotation group.

# First naive approach to the problem

- Consider the

  'space of all possible positions of these objects modulo the natural action of the rotation group'.

- That is a naive example of what is meant here by a moduli space.
- A naive test is whether the reported data of positions of the objects give the same point in this 'moduli space'.

# Methods

- The methods here are to introduce **moduli spaces** which are suitable for the problems, and which give reasonable tests.
- The tests are whether the values of certain functions described below are equal.
- There are several such spaces described below. One mathematical feature is that the first of these spaces is equivalent to a classical space given by Siegel upper $1/2$-space (described below).

# Three particles in orbit

- Start with three particles in orbit around a 'sun' with a fixed radius from the 'sun'. Are these objects rigidly attached ?
- Next, identify the space of

  ordered distinct triples of positions modulo the action of the rotation group.

# Three particles returned

- This ‘moduli space of three particles’ turns out to be classical.
- This space is one sheet of hyperbolic three space.
- This identification is via certain natural functions on the set of observations of the positions of the particles.
- Furthermore, analogous but more delicate 'moduli spaces' are developed to measure whether objects are rigidly attached.

# Configuration spaces

Let $M$ denote a topological space such as $m$-dimensional Euclidean space $\mathbb{R}^m$ or the $n$-sphere $S^n$. Let $k$ be a natural number. Define the configuration space of ordered $k$-tuples of distinct points in $M$

$$Conf(M, k)$$

as the subspace of the product $M^k$ given by

$$\{(m_1, \ldots, m_k) \mid m_i \neq m_j \text{ if } i \neq j\}.$$

## One example

- Consider three objects in a circular orbit of the same radii around a fixed center of mass. These particles, assumed to be ordered, are given by a point $(x, y, z)$ in the configuration space
$$Conf(S^2, 3).$$
- Furthermore, the rotation group $SO(3)$ acts on the $2$-sphere $S^2$ in the standard way, and thus acts diagonally on the configuration space. Namely, if
$$(x, y, z) \in Conf(S^2, 3),$$
and
$$\rho \in SO(3),$$
then
$$(\rho(x), \rho(y), \rho(z)) \in Conf(S^2, 3).$$

# One example continued

- If three objects are rigidly attached, then the data giving their positions at

  **any time t**

  must give identical elements in the 'orbit space' $Conf(S^2, 3)/SO(3)$.
- There is a homeomorphism

  $$\Theta : Conf(S^2, 3)/SO(3) \to \mathbb{H}^3_+$$

  where the target is one sheet of hyperbolic three-space.
- This function tests whether objects are rigidly attached and is described explicitly in the appendix.

# Two dimensional projections of orbits

- Start with particles moving around a fixed center of mass in $\mathbb{R}^3$ together with projection map to the first two coordinates

$$q : \mathbb{R}^3 \to \mathbb{R}^2.$$

  Each particle is assumed to be orbiting in a fixed plane.
- The purpose of this section is to retrieve the axis of rotation of an orbit from enough data points for the image of $q : \mathbb{R}^3 \to \mathbb{R}^2$ as follows. Assume the following.

# Two dimensional projections of orbits

- The particles are in an circular orbit (with modifications for elliptical orbits) around a center of mass located at the origin.
- The planes of the orbits are not orthogonal to the $xy$-plane.
- Then each particle is moving in a plane with normal vector $\vec{n} = (a, b, c)$ where the equation of the plane of the orbit is

$$ax + by + cz + d = 0$$

  for some values of $a, b, c, d$ which are not known.
- If the orbit is circular, the projection of this orbit will be either (i) an ellipse of positive area, or (ii) a 'degenerate' ellipse, a line segment with zero area. The second case (ii) cannot occur if the orbit is in a plane which is not orthogonal to the $xy$-plane.

# Moduli space of projected orbits

- The $2$-D projections of all possible (non-vertical) orbits gives another moduli space with good properties.
- Sufficient $2$-D projections determine the point in the moduli space.
- The axes of rotation of the particles are compared by constructing functions from the moduli space to real projective space. The axes of rotation agree if an only if these functions agree.
- Precise technical details are in the appendix.

# A language and context for legged robotic motion

- This section is based on joint work with Clark Haynes, and Dan Koditschek.

- The problem is to devise a practical, useful language for describing legged motion of certain robots.

# Setting

- The topological ingredients are a space of positions again, the so-called moment-angle complexes.
- The interiors of cells in a cell decomposition gives 'gait states' for the legs of a legged motion.
- The purpose here is to describe the possible gait states in terms of Young diagrams, then to construct vector fields on these interiors.
- Further applications are intended.

## Definition of moment-angle complex

- The moment-angle complex determined by $(X, A)$ and $K$ denoted $Z(K; (X, A))$ is defined as follows:
- For every $\sigma$ in $K$, let

$$D(\sigma) = \prod_{i=1}^{m} Y_i, \quad \text{where} \quad Y_i = \begin{cases} X_i & \text{if} \quad i \in \sigma \\ A_i & \text{if} \quad i \in [m] - \sigma. \end{cases}$$

with $D(\emptyset) = A_1 \times \cdots \times A_m$.

- The generalized moment-angle complex is

$$Z(K; (X, A)) = \bigcup_{\sigma \in K} D(\sigma) = \text{colim } D(\sigma).$$

# Spaces of legs

- Let

$$Legs(m,q)$$

denote the space of ordered $m$-tuples in a circle $S^1$ with at most $q$ "off of the ground". That means at most $q$ of the coordinates are in the open upper hemisphere $U^+$ of the circle, the complement of the closed lower hemisphere $E_-$.

- A mathematical starting point is as follows: If $q = 2$,

$$Legs(m,2) = Z(K;(S^1,E_-))$$

where $K$ is the complete graph with $m$ vertices. If $q \geq 2$ and $K = \Delta[m-1]_{q-1}$, the $(q-1)$-skeleton of the $(m-1)$-simplex, then

$$Legs(m,q) = Z(K;(S^1,E_-)).$$

# Enumerating 'gait states' in spaces of legs

- This section provides a language which describes 'gait states'.
- A convenient form of this language is in terms of **Young diagrams** a construction originally invented to study the representation theory of the symmetric groups.

# Topological interlude: Decompositions of moment-angle complexes

- Moment-angle complexes are fragile in the sense that they decompose into 'simple' pieces after suspending: A. Bahri, M. Bendersky, S. Gitler, and the author: P.N.A.S., 29 July 2009.

- One of their main useful features arises from features of the Whitehead product.

# Young diagrams

- A **Young diagram or Ferrers diagram** is an array of $n+1$ boxes in $k+1$ rows. Filling in these boxes with all of the integers $1, 2, \ldots, n, n+1$ gives all of the **Young tableaux**.
- These Young tableaux index cells in $T^n$ ( where $n+1$ boxes is not a misprint) . in a way that is intuitively meaningful as well as a setting to compute.

# Young diagrams, more formally

- The set of 'filled in Young diagrams', 'Young tableaux' $Y(n+1, k+1)$ is the set of arrays

$$[a_{i,j}] = \begin{array}{ccc} a_{1,1} & \cdots & a_{1,j_1} \\ a_{2,1} & \cdots & a_{2,j_2} \\ \cdots & \cdots & \cdots \\ a_{k+1,1} & \cdots & a_{k+1,j_{k+1}} \end{array}$$

with $n+1$ entries given by the set of all of the integers between $1$, and $n+1$.

# Young diagrams, more formally

- The diagrams

$$[a_{i,j}]$$

are also specified by their rows

$$(R_i)$$

with notation

$$[a_{i,j}] = (R_i)$$

where

$$1 \leq i \leq k+1.$$

# Face operations in Young diagrams

- Define

$$d_i([a_{i,j}]) = d_i(R_t) = (S_q) \in Y(n+1,k) \text{ with } 1 \leq i \leq k$$

where

$$S_q = \begin{cases} R_q & \text{if } q < i, \\ [R_q|R_{q+1}] & \text{if } q = i < k+1, \\ R_{q+1} & \text{if } q > i. \end{cases}$$

# Face operations continued

- Define

$$d_{k+1}([a_{i,j}]) = d_i(R_t) = (S_q) \in Y(n+1,k)$$

where

$$S_q = \begin{cases} [R_1|R_{k+1}] & \text{if } q = 1, \\ R_q & \text{if } 1 < q \leq k. \end{cases}$$

# Cyclic permutations and face operations in Young diagrams

- Recall that $C_{k+1}$ denotes the cyclic group generated by the $(k+1)$-cycle $t_{k+1} = (1, 2, \cdots, k+1)$.
- The operations

$$d_i : Y(n+1, k+1) \to Y(n+1, k),$$

and

$$t_{k+1} : Y(n+1, k+1) \to Y(n+1, k+1)$$

satisfy the identities

$$d_i t_{k+1} = t_k d_{i-1} \text{ if } 1 \leq i \leq k,$$

and

$$d_0 t_{k+1} = d_k.$$

## ‘Local flows’:

- Each gait state corresponds to the interior of a cell in the moment-angle complex.
- Flows are defined on each cell to prescribe motion.

# Conclusion

- The 'gait states' in a product of circles is enumerated by the 'cyclic-Delta' set structure given above.
- The open cells correspond to all possible 'gait states'.
- The 'motions' of the boxes in the Young diagrams via the natural action of the cyclic group corresponds to motions of legs.

# Statement of the problem of counting with sensors

- joint work with Doug Cochran, Bill Moran and Stephen Howard
- Given $M$ sensors $S_1, \cdots, S_M$ in the plane, or $\mathbf{R}^n$ for which each $S_i$ reports $x_i$ elements, how many such elements are in the union ?
- The problem is not well-posed. There are different solutions which can be framed as follows which include both static and dynamic settings.
- Find the most likely minimum number in the union.
- Find the most likely number in the union.

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# Counting with sensors: A problem concerning finite simplicial complexes

The intersections of the sensor information is indexed by a simplicial complex as follows.

- Let $K$ denote a simplicial complex with $m$ vertices.
- A convex polytope $P(K)$ in $\mathbb{R}^m$ is defined next.
- Find the vertices of $P(K)$.

# The convex polytope $P(K)$

- A simplicial complex $K$ with $m$ vertices is a subset of the power set $\{1, 2, \ldots, m\}$.
- If $\sigma$ is in $K$, and $\tau \subset \sigma$, then $\tau$ is in $K$.
- The convex polytope

$$P(K)$$

is defined to be the subset of $\mathbf{R}^m$ given by the points

$$(x_1, \ldots, x_m)$$

such that

1. $0 \leq x_i$ for all $i$, and
2. $x_{i_1} + \cdots + x_{i_t} \leq 1$ for every face $\sigma = \{i_1, \ldots, i_t\}$ in $K$.

# The convex polytope $P(K)$ and estimates

- The estimates given here for the most likely maximum and minimum are given by functions defined on the simplical complex $K$ as well as the convex polytope $P(K)$.

- Find the vertices of $P(K)$.

# Appendix

- Explicit tracking formulas for the moduli spaces above are given.