Braid groups of graphs & robot motion planning

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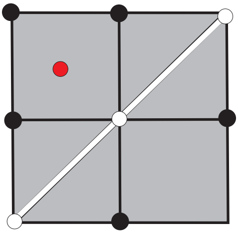
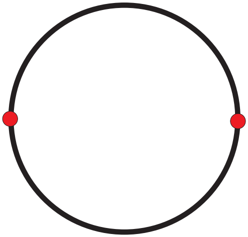
Two robots in a segment

A position of two ordered robots is coordinates x, y of a single point in the space $\{(x, y) \in [-1, 1]^2 \,:\, x \neq y\}$.

OC([-1,1],2):



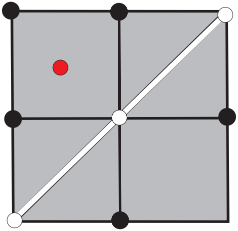


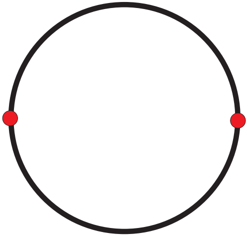
Robots in a graph

Def : the ordered configuration space of n distinct robots in a graph G is

$$\mathrm{OC}(G,n) = \{(x_1,\ldots,x_n) \in G^n \ :\ x_i \neq x_j\}$$

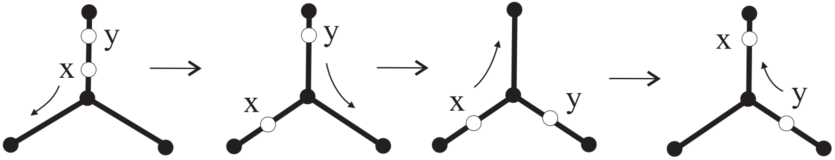
OC([-1,1],2):





n robots in a tripod

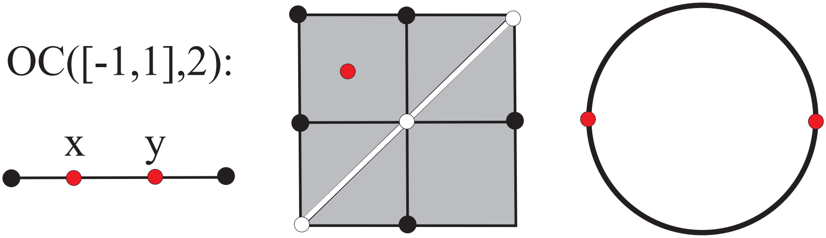
The configuration space of n robots in any connected graph with an essential vertex of degree ≥ 3 is connected.



Simplifying the space

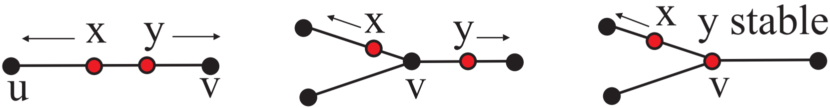
Non-compact $\mathrm{OC}(G, n)$ deformation retracts to a small compact subspace.

$\mathrm{OC}([-1, 1], 2) \sim \{**\}$, $\mathrm{OC}(S^1, 2) \sim S^1$



Discrete spaces

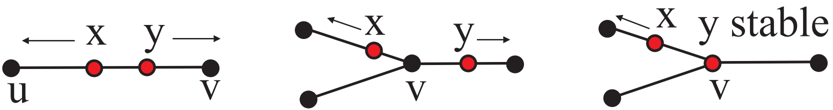
The discrete space $\mathrm{OD}(G, n)$ consists of $(x_1, \ldots, x_n)$ with $\mathrm{supp}(x_i) \cap \mathrm{supp}(x_j) = \emptyset$, $\mathrm{supp}(x) = x$ if x is a vertex, otherwise $\mathrm{supp}(x)$ is the (closed) edge $\ni x$.



Repelling robots

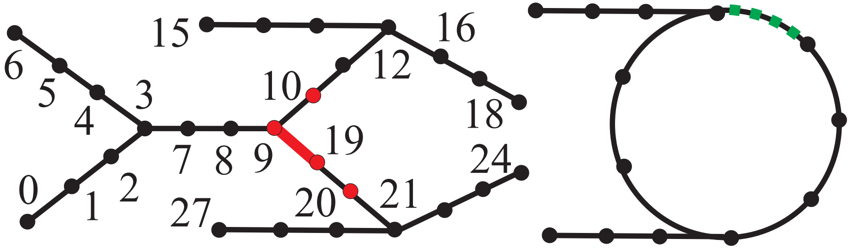
If G has no loops and multiple edges then $\mathrm{OC}(G, 2)$ def. retracts to $\mathrm{OD}(G, 2)$.

Two robots are moving away from each other until $\mathrm{supp}(x) \cap \mathrm{supp}(y) = \emptyset$.



Sufficient subdivisions

Th (Abrams): $\mathrm{OC}(G, n)$ deformation retracts to $\mathrm{OD}(G, n)$ if any path between non-trivial vertices has $\geq n + 1$ edges.



Braid groups of graphs

help understand the homotopy type of a configuration space of n robots.

Def : for a connected graph G with an essential vertex, the braid groups are

$$\mathrm{B}(G, n) = \pi_1(\mathrm{UC}(G, n)) \cong \pi_1(\mathrm{UD}(G, n)),$$

$$\mathrm{P}(G, n) = \pi_1(\mathrm{OC}(G, n)) \cong \pi_1(\mathrm{OD}(G, n))$$

if edges of G are sufficiently subdivided.

Computational methods

Braid groups of graphs can be found by

- identifying a configuration space
- splitting as a graph of groups
- discrete Morse theory of R Forman
- the Seifert – Van Kampen theorem

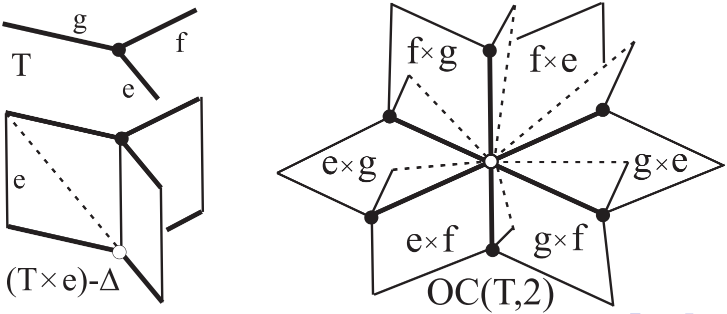
Kuratowski graphs

$K_5, K_{3,3}$ are the only graphs G such that $\mathrm{OC}(G, 2)$ deform. retracts to a surface.

$\mathrm{OD}(K_5, 2) = \cup$ 10 triangular tubes $\approx \Sigma_6$,
$\mathrm{OD}(K_{3,3}, 2) = \cup$ 9 square tubes $\approx \Sigma_4$.

Essential subcomplex

Th (Ghrist): if $G \not\approx S^1$ has k essential vertices (deg $\geq$ 3) then $\mathrm{OC}(G, n)$ def. retracts to a k-dimensional complex.

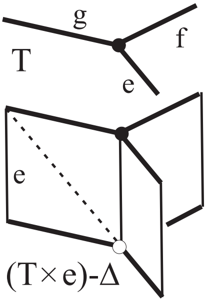


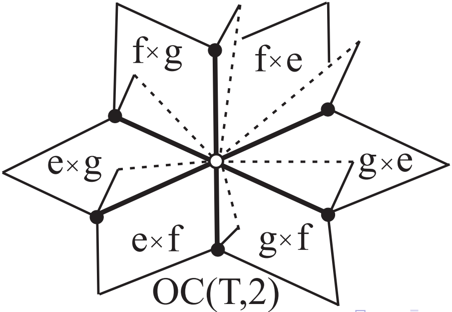
Braid groups of stars

Corollary: $\mathrm{P}(T_k, n)$ is the free group F_r,

$r = 1 + (kn - 2n - k + 1)(n + k - 2)!/(k - 1)!$,

e.g. $\mathrm{P}(T_k, 2) \cong F_{k^2 - 3k + 1}$, $\mathrm{P}(T, 2) \cong \mathbb{Z}$.





A discrete Morse theory

The approach by D Farley, L Sabalka gives presentations of $\mathrm{B}(G, n)$, where generators are critical 1-cells, relations are boundary words of critical 2-cells.

The full structure of a graph is needed. *Open problem:* for a planar graph G, is $\mathrm{B}(G, 2)$ a commutator-relator group?

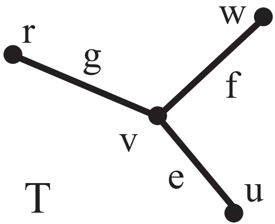
A recursive computation

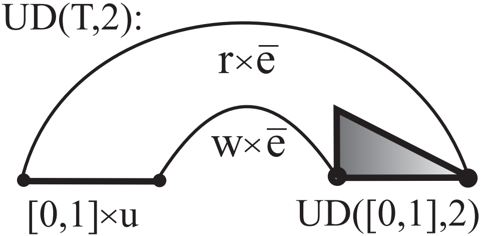
Using Seifert – van Kampen's theorem, $\mathrm{B}(G, n)$ and $\mathrm{P}(G, n)$ can be computed step-by-step for a growing graph G:

- adding a hanging edge
- stretching a hanging edge
- adding an edge creating cycles

Fixing a robot in a tripod

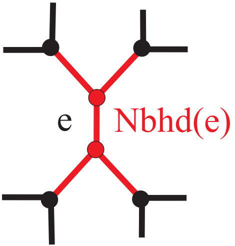
If $x, y \notin e$ then (say) $y = u$, $x \in [0, 1]$ or $(x, y) \in \mathrm{UD}([0, 1], 2)$. Then $\mathrm{UD}(T, 2) = \mathrm{UD}([0, 1], 2) \cup ([0, 1] \times u) \cup (\{w, r\} \times \bar{e})$.

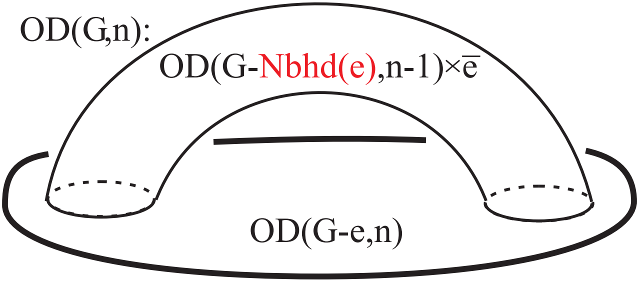




A recursive construction

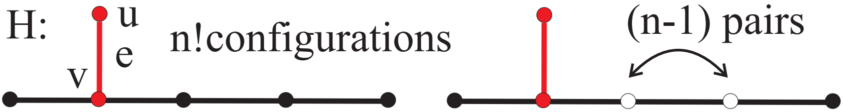
$\mathrm{OD}(G, n)$ is obtained from $\mathrm{OD}(G - e, n)$ by attaching several cylinders over the subspace $\mathrm{OD}(G - \mathrm{Nbhd}(e), n - 1)$.





A free pure braid group

Let $H = [0,1] \cup e \cup u$ have $n+2$ vertices. Since $\mathrm{OD}([0,1], n) \sim n!$ then $\mathrm{OD}(H, n) \sim$ a graph with $n!$ vertices and $n!(n-1)$ edges, hence $\mathrm{P}(H, n) \cong F_{1+n!(n-2)}$.



Generators as motions

The fundamental group is computed by adding motions along new edges.

We need a planning algorithm joining given configurations of n robots.

Our algorithm has complexity $O(n^2 l)$, where $l = \#$edges and $n = \#$robots.

An extreme robot, neighbour

A robot x_i is extreme in $(x_1, \ldots, x_n)$ if the other robots are in a one component of $G - \{x_i\}$, e.g. 2 extreme robots on $[0, 1]$. Find an extreme one: $O(nl)$ operations.

x_j is a neighbour of x_i if a shortest path from x_i to x_j has minimum $\#$ edges over paths from x_i to x_k: a collision free path.

Motion planning algorithm

All robots are at vertices, may be deg 2.

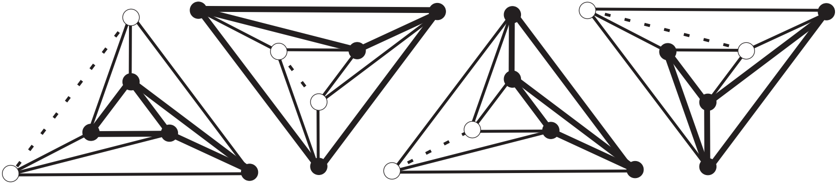
Step 1: find an extreme robot (say y_n) among initial x_i and final y_i positions.

Step 2: find a shortest path from y_n to its neighbour (say x_n), move x_n to y_n keeping the robots $x_1, \dots, x_{n-1}$ fixed.

Step 3: repeat $n-1$ times Steps 1–2.

Light planar graphs

A planar connected graph G is light if any cycle $C \subset G$ has an open edge h: all cycles from $G - \bar{h}$ do not meet C.



A non-light graph with four choices of $\bar{h}$.

Commutator-relator groups

Corollary: any light planar graph G is constructed from a tree T by adding $(*)$ edges $G_1 = T \cup h_1, \ldots, G = G_{m-1} \cup h_m$. If $G_j - \mathrm{Nbhd}(h_j)$ has k_j components then $\mathrm{B}(G, 2)$ has a presentation with $\sum k_j + \sum(\deg v - 1)(\deg v - 2)/2$ gen. and commutator relations: motions of 2 robots along disjoint cycles commute.

Summary

- based on the simpler classical Seifert – van Kampen theorem
- understood: light planar graphs with commutator-relator groups
- generators are expressed as collision free motions of robots
- suitable for real-time adding edges