A Hierarcy of Directed Structures.

Lisbeth Fajstrup

Department of Mathematical Sciences, University of Aalborg

Overview

- d-Top, the category of directed spaces.
- Lattice structure on d-Top
- Relative directed structure
- The main example: The n-cube minus a forbidden region.
- Conclusion, further work.

Definition

Objects of **d-Top** are d-Spaces: $(X, \vec{P}(X))$ where $X \in \textbf{Top}$, $\vec{P}(X) \subseteq X^I$, the dipaths. $\vec{P}(X)$ is

- closed under concatenation,
- contains the constant paths
- closed under subpath, i.e., composition with $f : I \to I$ increasing but not necessarily surjective.

A d-map $f : X \to Y$ is a continuous map satisfying $\gamma \in \vec{P}(X) \Rightarrow f \circ \gamma \in \vec{P}(Y)$

Hierarchy of d-structures

Let X be a topological space.
$\mathcal{P}(X) \subset 2^{X^I}$ the set of possible d-structures on X.
Partial order on $\mathcal{P}(X)$:
$\vec{P}(X) \leq \vec{Q}(X)$ if $id : (X, \vec{P}(X)) \to (X, \vec{Q}(X))$ is a d-map.

Extension: If τ, σ are topologies on the underlying set of X and $(X, \tau, \vec{P})$ and $(X, \sigma, \vec{Q})$ are d-spaces
$(X, \tau, \vec{P}) \leq (X, \sigma, \vec{Q})$ if the identity map is a d-map.

Lattice properties

Definition

For $\vec{P}, \vec{Q} \in \mathcal{P}(X)$ define

- Join $\vec{P} \vee \vec{Q} = (\vec{P} \cup \vec{Q})^*$ where $-^*$ is closure under concatenation and reparametrization.
- Meet $\vec{P} \wedge \vec{Q} = \vec{P} \cap \vec{Q}$
- Bottom: $\perp$ is the set of constant paths.
- Top: $\top = X^I$

Lattice properties

Proposition

$(\mathcal{P}(X), \leq)$ is a complete distributive lattice.

Completeness: For a subset $\vec{P}_j, j \in J$

- The join $\bigvee_{j\in J} \vec{P}_j = (\bigcup_{j\in J} \vec{P}_j)^*$ where $-^*$ is closure under finite concatenation.
- Meet is the intersection.

Sketch of proof

Distributivity: Need

$$\vec{P} \vee (\vec{Q} \wedge \vec{R}) = (\vec{P} \vee \vec{Q}) \wedge (\vec{P} \vee \vec{R})$$

$$i.e. \ (\vec{P} \cup (\vec{Q} \cap \vec{R}))^* = (\vec{P} \cup \vec{Q})^* \cap (\vec{P} \cup \vec{R})^*$$

$\subseteq$: Let $\gamma = (\gamma_1 \star \gamma_2 \cdots \gamma_n) \circ \alpha$, where $\gamma_{2i} \in \vec{P}$ and $\gamma_{2i+1} \in \vec{Q} \cap \vec{R}$.
Then $\gamma \in \vec{P} \vee \vec{Q}$ and $\gamma \in \vec{P} \vee \vec{R}$.
$\supseteq$: $\mu = (\mu_1 \star \mu_2 \cdots \star \mu_r) \circ \alpha$, $\mu_{2k} \in \vec{Q}$, $\mu_{2k+1} \in \vec{P}$
and $\mu = (\eta_1 \star \eta_2 \cdots \star \eta_l) \circ \beta$ where $\eta_{2i} \in \vec{R}$, $\eta_{2i+1} \in \vec{P}$.
We may assume α and β are piecewise linear (changing the subintervals, where μ_j and η_m are defined.)
Use a common subdivision $0 = t_0 \leq t_1 \leq t_2 \ldots \leq t_m = 1$, of $[0,1]$ s.t. $\mu_{[t_j, t_{j+1}]}$ is a restriction of some μ_r and some η_s. Then $\mu_{[t_j, t_{j+1}]}$ is in either $\vec{P} \cap \vec{P}$, $\vec{P} \cap \vec{R}$, $\vec{Q} \cap \vec{P}$ or $\vec{Q} \cap \vec{R}$ and we are done.

Complements

$$\neg\vec{P} = \{\gamma \in X^I | \forall t_1 \leq t_2 \in I : \gamma_{[t_1,t_2]} \in \vec{P} \Leftrightarrow \gamma([t_1,t_2]) = \gamma(t_1)\}$$

$\neg\vec{P}$ is a pseudo complement:
$\neg\vec{P} \wedge \vec{P} = \perp$ but $\neg\vec{P} \vee \vec{P} \neq \top$ (Ex. $(t, t\sin(1/t))$)

$$\neg\neg P = \{\gamma \in X^I | \forall t_1 < t_2 \in I \exists t_1', t_2' : t_1 \leq t_1' < t_2' \leq t_2, \gamma|_{[t_1',t_2']} \in \vec{P}\}$$

$\vec{P} \subsetneq \neg\neg\vec{P}$ (Ex: Fractal saw tooth curve)

Complements

Theorem

Let $\vec{P}$ be closed in X^I wrt. the co-topology. Let $-^{\omega}$ be closure under infinite concatenation (and reparametrization and subpath). Then

$$(\vec{P} \vee \neg\vec{P})^{\omega} = X^I$$

I.e. $\neg\vec{P}$ is a complement in the lattice structure where join implies infinite concatenation. Perhaps a d-structure should be closed under $(-)^{\omega}$

BUT $\vec{P}$ closed $\Rightarrow \neg P$ is *not* closed.

Forbidden areas and d-structures

An adjunction $\mu : 2^X \rightleftarrows \mathcal{P}(X) : \nu$
$F \subset X$

$$\mu(F) = \{\gamma \in X^I | \gamma(I) \cap F \neq \emptyset \Rightarrow \gamma \text{constant}\}$$

$\mu(F)$ is the maximal set of paths which do not enter F except for the constant paths. For $\vec{Q} \in \mathcal{P}(X)$

$$\nu(\vec{Q}) = \{x \in X | \vec{Q}(x, -) = \vec{Q}(-, x) = \star\}$$

$\nu(\vec{Q})$ is the set af fixed points.

$$F_1 \subseteq F_2 \Rightarrow \mu(F_1) \supseteq \mu(F_2)$$

$$\vec{Q}_1 \subseteq \vec{Q}_2 \Rightarrow \nu(\vec{Q}_1) \supseteq \nu(\vec{Q}_2)$$

μ and ν are *decreasing*

Galois connection.

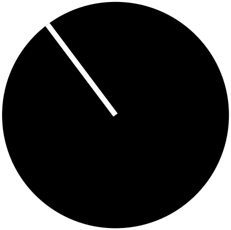
Theorem

μ is a right (upper) adjoint to ν

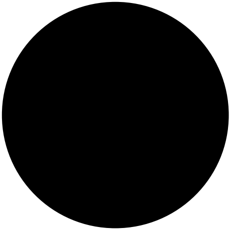
$$\mu : (2^X, inclusion)^{op} \rightleftarrows (\mathcal{P}(X), inclusion) : \nu$$

(μ, ν) is a Galois connection.

Proof: $\vec{Q} \subseteq \mu(F) \Leftrightarrow \nu(\vec{Q}) \supseteq F$ - just check it...



Forbidden area F



$\nu(\mu(F))$

d-structures on the n-cube.

Orders on the interval: $\vec{I}$: standard order
$\overleftrightarrow{I}$ $t \leq s$ for *all* $t, s \in I$
Induces partial orders - hence d-structures on I^n:

$$\vec{I}^k \times \overleftrightarrow{I}^{n-k}$$

Hierarchy induced by $\vec{I} \leq \overleftrightarrow{I}$ as d-structures.
Given $X = \vec{I}^n \setminus F$ - model for PV-programs.
Study

$$\vec{I} \times \vec{I} \times \cdots \times \overleftrightarrow{I} \times \cdots \times \vec{I} \setminus F$$

The partial order is relaxed in the j'th coordinate.
$(x_1, \ldots, x_n) \leq (y_1, \ldots y_n)$ if $x_k \leq y_k$ for all $k \neq j$.

Applications

Reversible Computing.

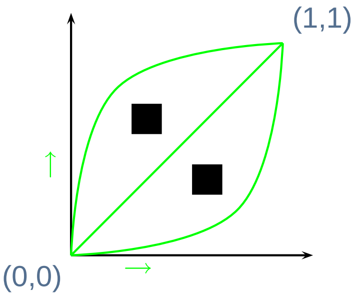
- Vincent Danos, Jean Krivine: *Reversible Communicating Systems.* RCCS - reversible CCS.
- V. Danos, J. Krivine, P. Sobociński *General Reversibility*.

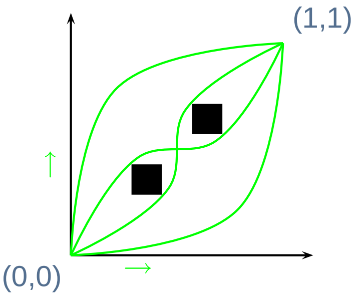
Questions:

- Does reversing give more equivalences of dipaths - i.e. easier to verify.
- More (equivalence classes of) dipaths - i.e., a program with more possible outputs. Higher expressiveness.

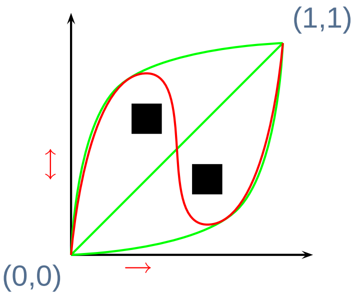
A geometric model for backtracking ?

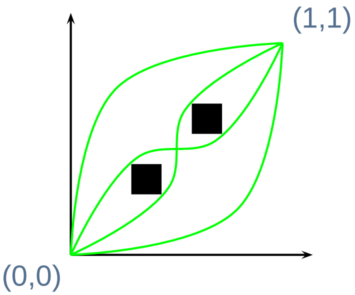
2d examples



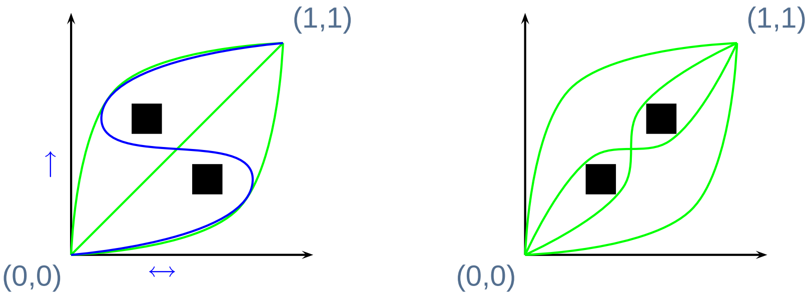


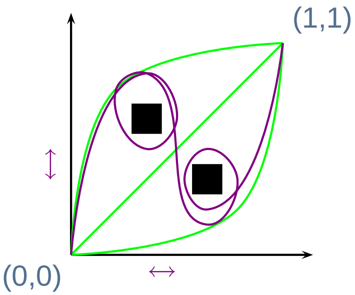
2d examples

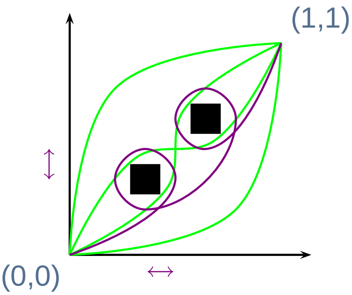




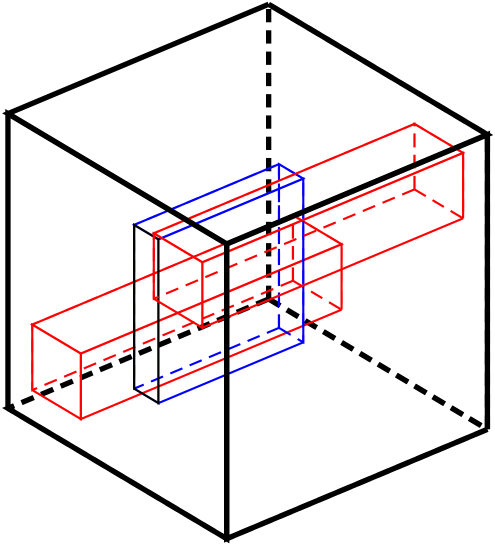
2d examples



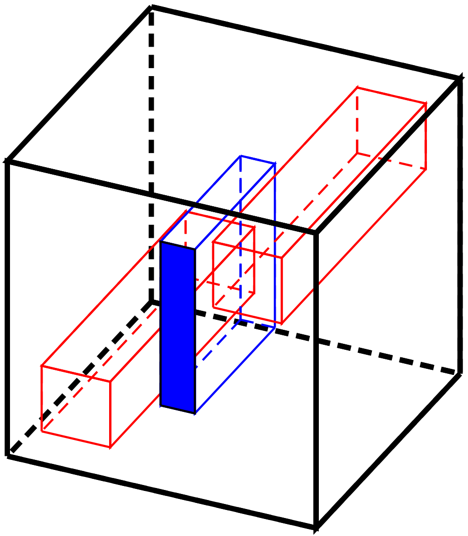

(1,1)
(0,0)


(1,1)
(0,0)

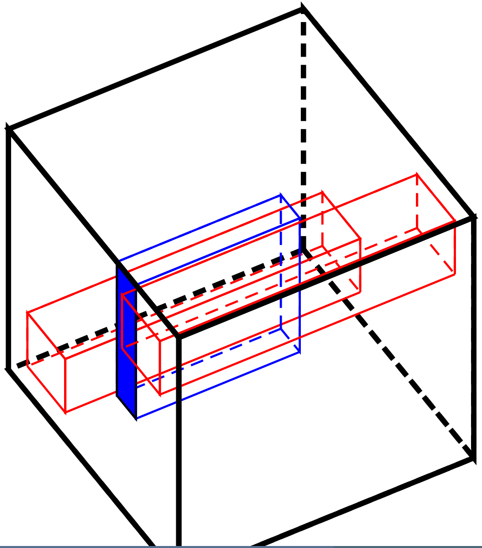
3d examples



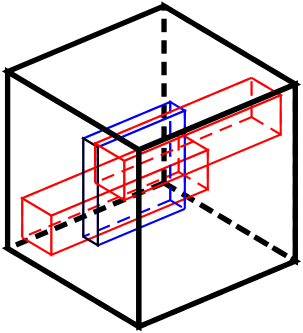
3d examples



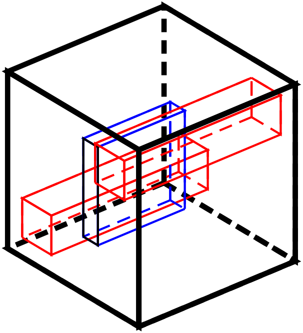
3d examples

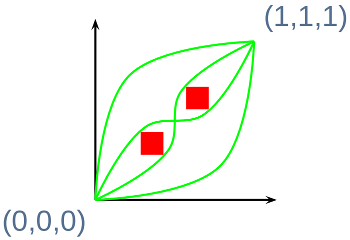


Dipaths in 3D-example

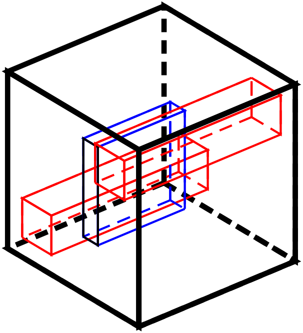


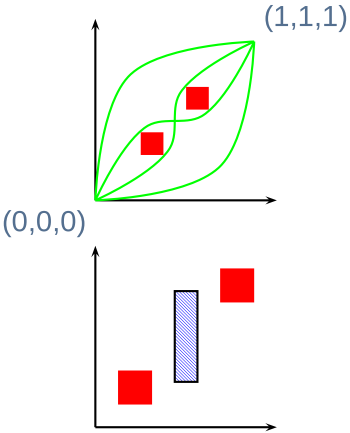
Dipaths in 3D-example



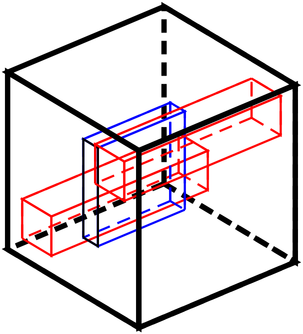


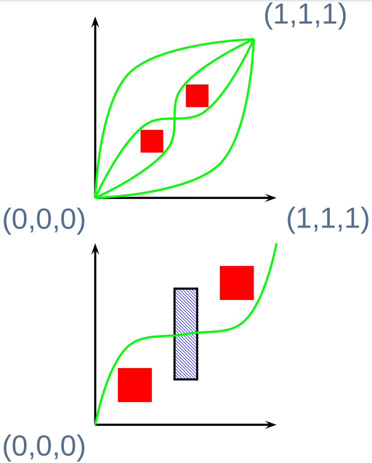
Dipaths in 3D-example





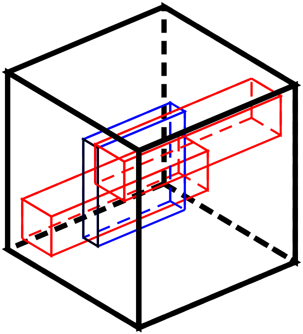
Dipaths in 3D-example

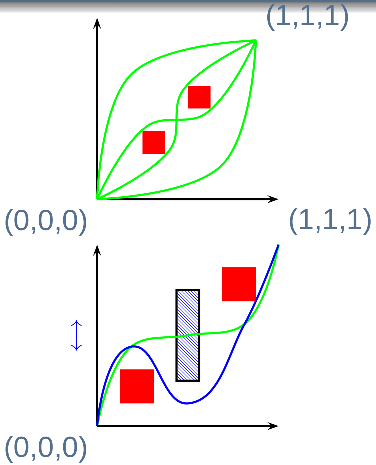




Two non-equivalent dipaths. In front of and behind blue cube

Dipaths in 3D-example

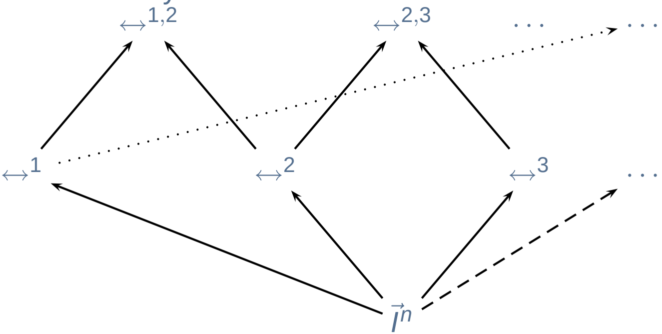




Two non-equivalent dipaths. In front of and behind blue cube
Now equivalent, when releasing third coordinate.

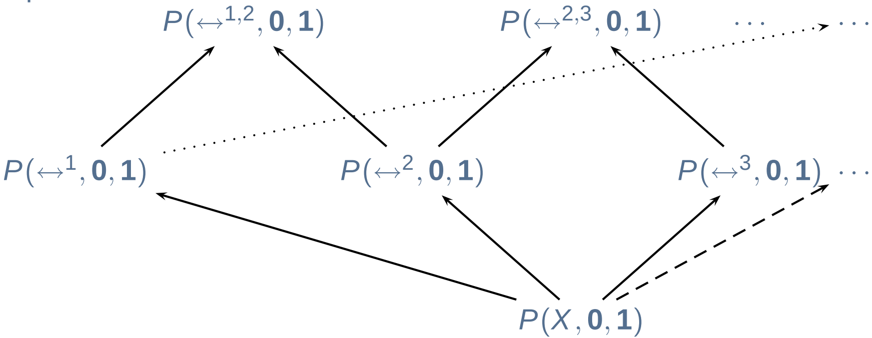
Path- and trace- spaces

Let $X = \vec{I}^n \setminus F$. $\leftrightarrow^j$ is X with d-structure relaxing the j'th coordinate.
The hierarchy of d-structures:



Induces a hierarchy - via inclusion - of dipath spaces and trace spaces.

$P(X, x_0, x_1)$ dipaths from x_0 to x_1. $P(\leftrightarrow^{i,j}, \mathbf{0}, \mathbf{1})$ the set of dipaths wrt. $\leftrightarrow^{i,j}$.



Relative homology

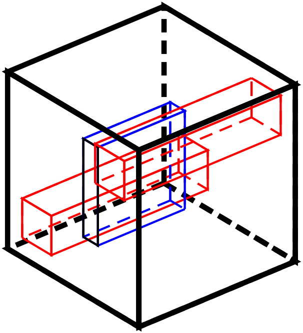
$P(X, \mathbf{0}, \mathbf{1}) \subset P(\leftrightarrow^i, \mathbf{0}, \mathbf{1})$. Hence

$$\to H_{k+1}(P(X)) \to H_{k+1}(P(\leftrightarrow^i)) \to H_{k+1}(P(\leftrightarrow^i), P(X)) \to H_k(P(X) \to$$

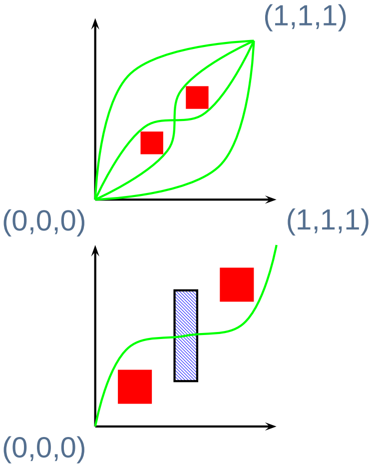
And for the triple $P(X, \mathbf{0}, \mathbf{1}) \subset P(\leftrightarrow^i, \mathbf{0}, \mathbf{1}) \subset P(\leftrightarrow^{i,j}, \mathbf{0}, \mathbf{1})$
$P(X, \mathbf{0}, \mathbf{1}) \subset P(\leftrightarrow^i, \mathbf{0}, \mathbf{1})$. Hence

$$\to H_{k+1}(\leftrightarrow^j, P(X)) \to H_{k+1}(P(\leftrightarrow^{i,j}, P(X))) \to H_{k+1}(P(\leftrightarrow^{i,j}), P(\leftrightarrow^i)) -$$

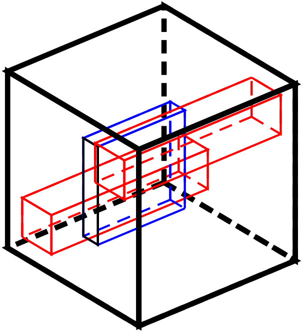
Dihomotopy classes are connected components of the path space, i.e., seen from H_0.

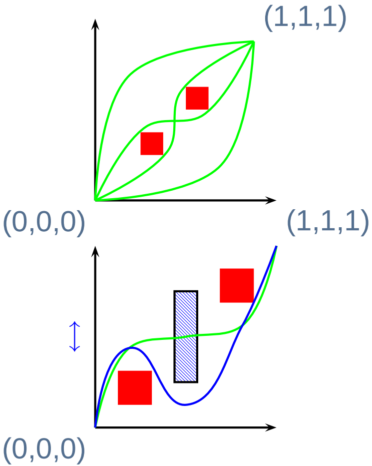


$P(X, \mathbf{0}, \mathbf{1}) = S^1 \sqcup_{k=1}^{4} *_k$



Back to examples

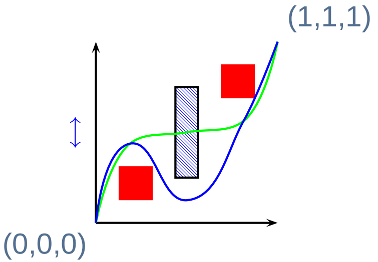




$$P(X, \mathbf{0}, \mathbf{1}) = S^1 \sqcup_{k=1}^{4} *k$$
$$P(\leftrightarrow^3) = \sqcup_{l=1}^{4} S^1_l$$

Homology sequence for the example

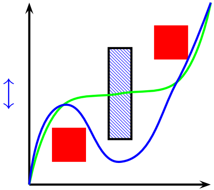
$$\cdots H_1(P(X)) \to H_1(P(\leftrightarrow^3)) \to H_1(P(\leftrightarrow^3), P(X))$$
$$\to H_0(P(X)) \to H_0(P(\leftrightarrow^3)) \to$$



Homology sequence for the example

$$\cdots \mathbb{Z} \to \mathbb{Z}^4 \to H_1(P(\leftrightarrow^3), P(X)) \leftrightarrow$$
$$\mathbb{Z}^5 \to \mathbb{Z}^4 \to H_0(P(\leftrightarrow^3), P(X)) = 0$$

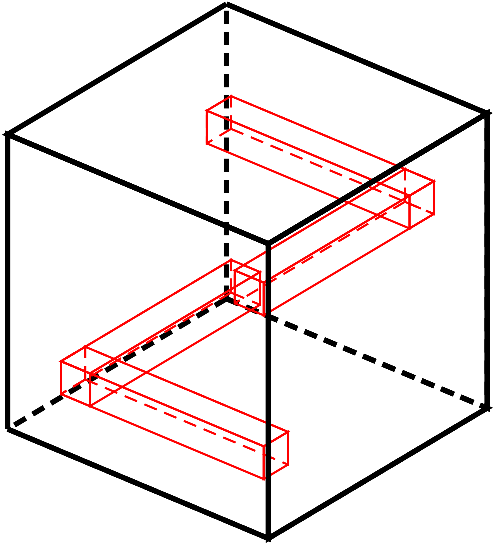
(1,1,1)



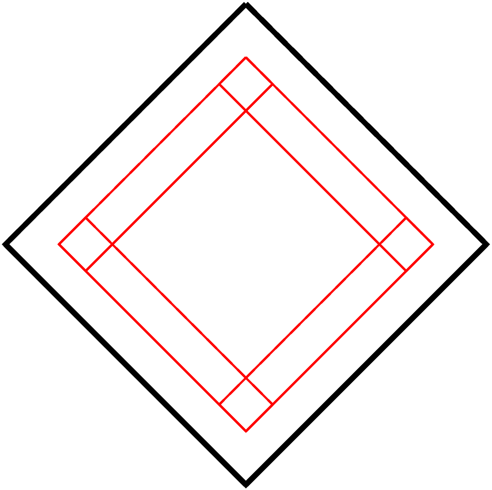
(0,0,0)

The *obstruction* to a homotopy between front and back of the blue cube is along the 3'rd axis.

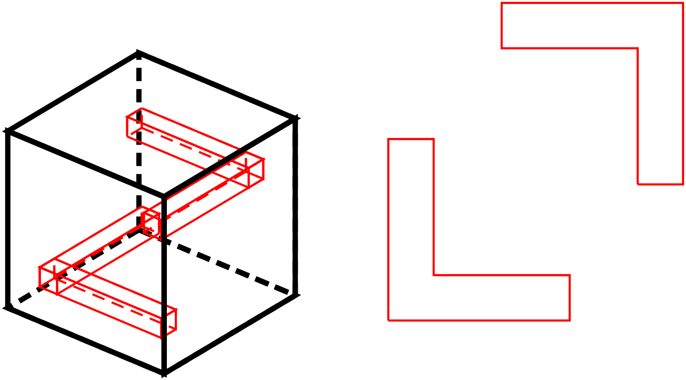
Two wedges



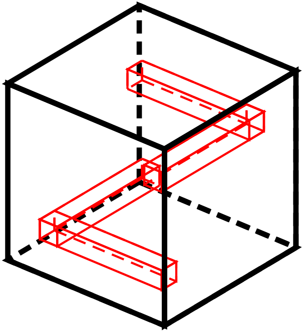
Two wedges

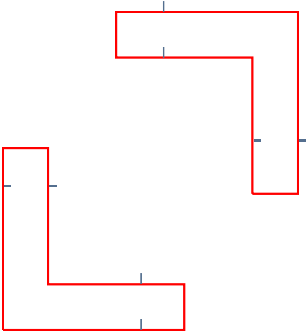


Two wedges

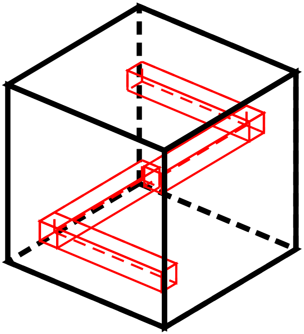


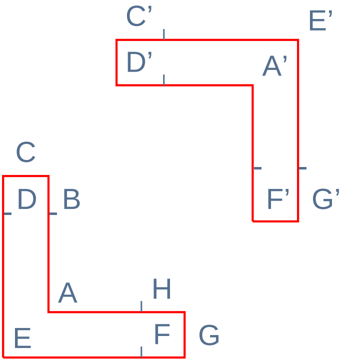
Two wedges



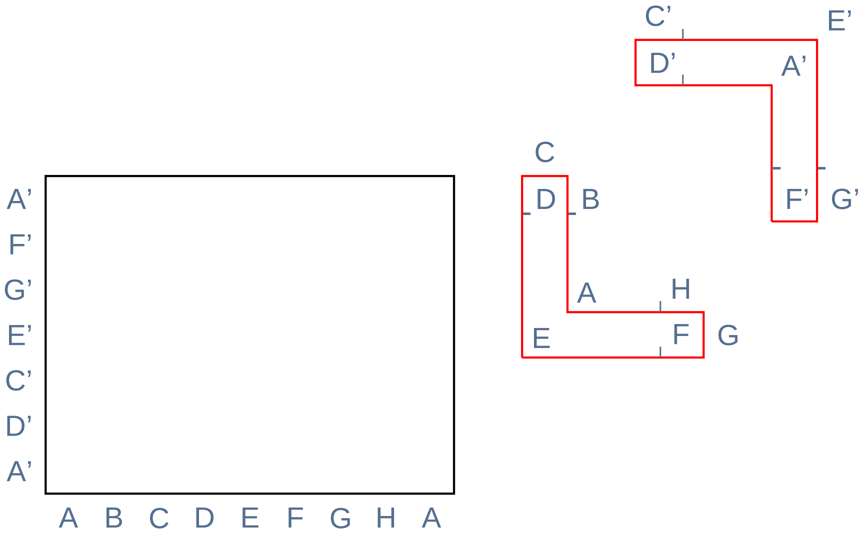


Two wedges

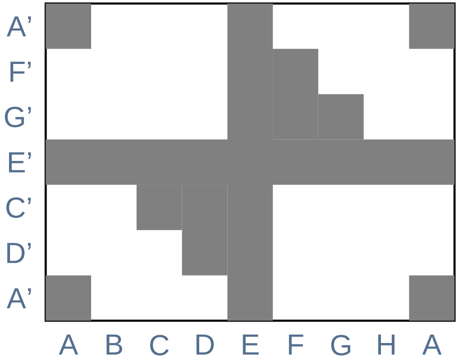


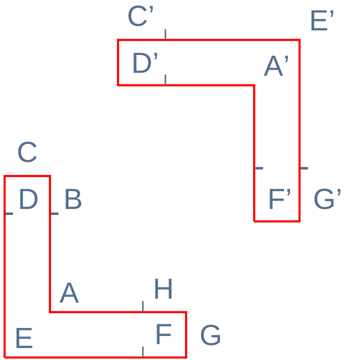


Reachability analysis

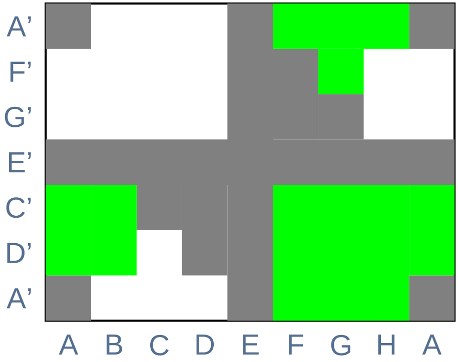


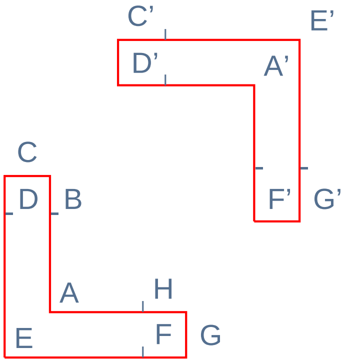
Reachability analysis





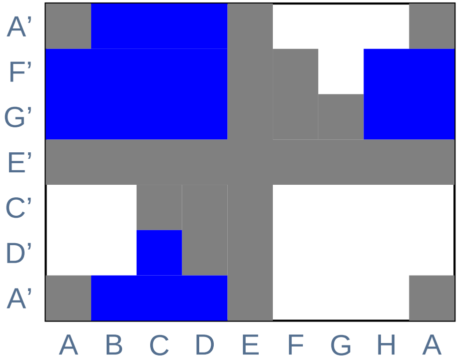
Reachability analysis

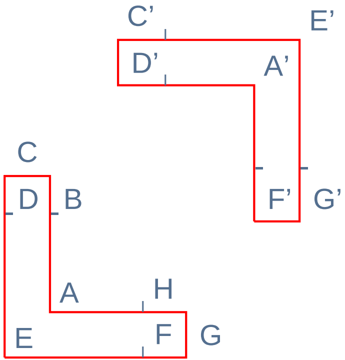




$\leftrightarrow$ A punctured torus

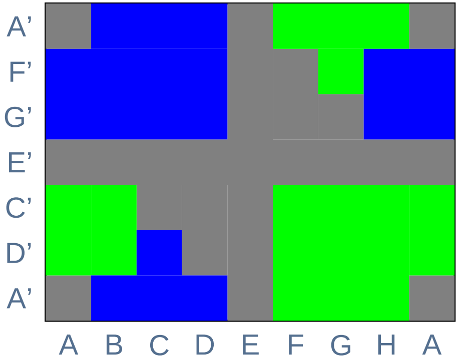
Reachability analysis

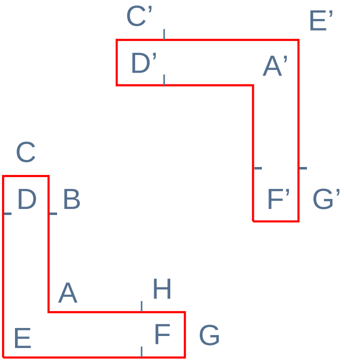




↕ A punctured torus

Reachability analysis

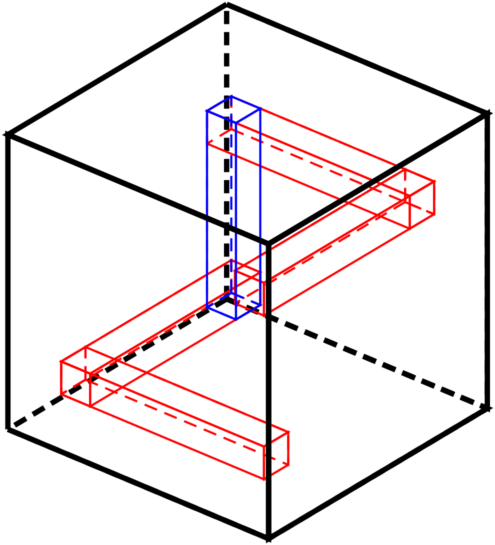




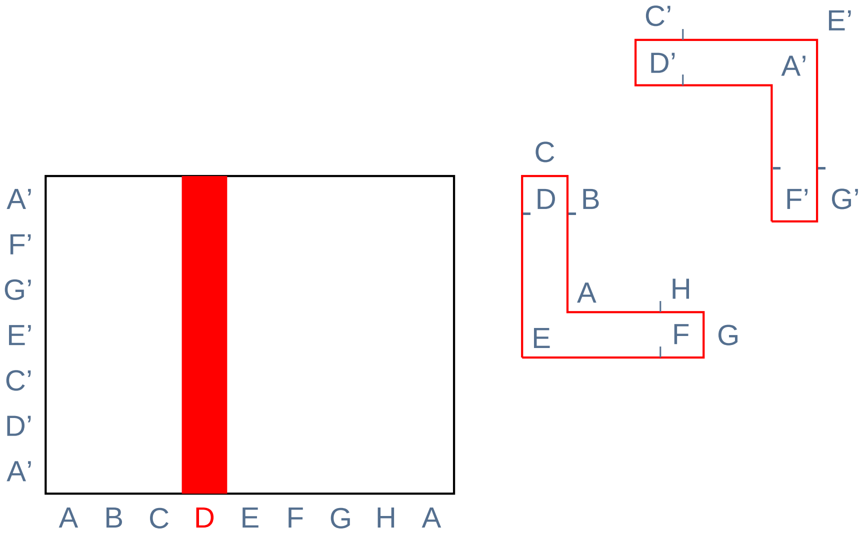
↔ A punctured torus

↕ A punctured torus

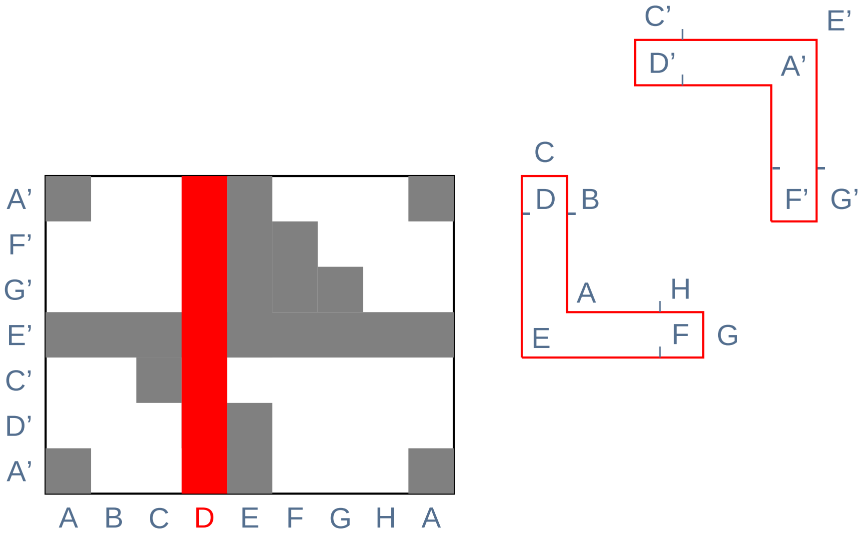
Two wedges - connected



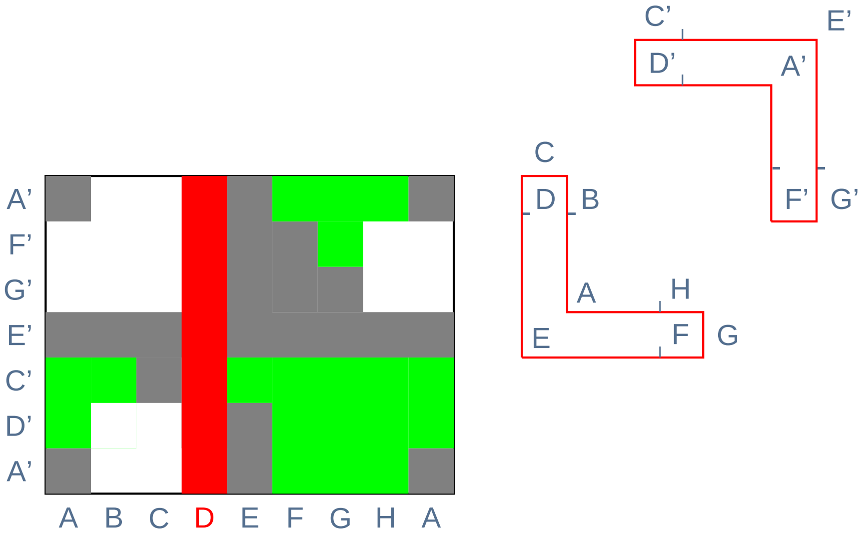
Reachability analysis



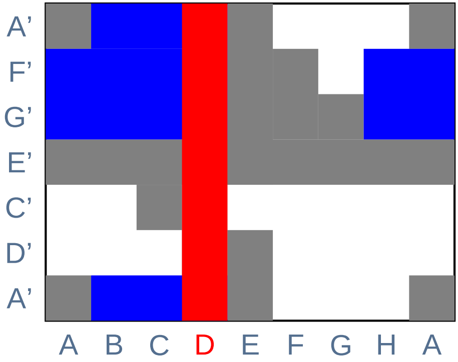
Reachability analysis

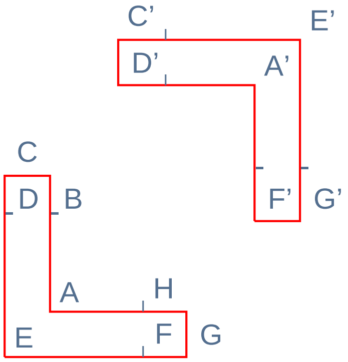


Reachability analysis

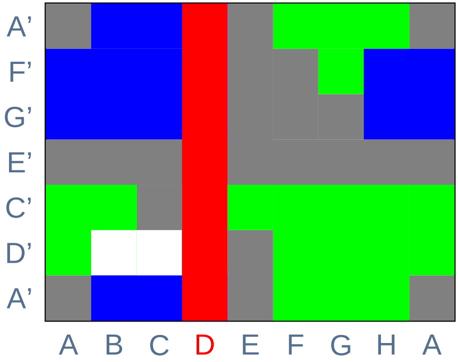


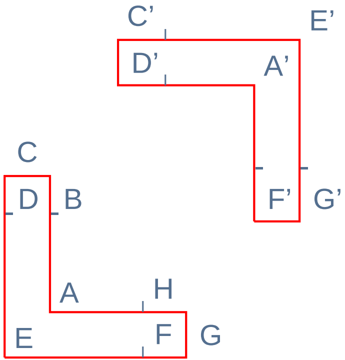
Reachability analysis





Reachability analysis





A spiral

Let the forbidden area be a spiral/helix along the z-axis (several wedges connected). Then the directed paths through the spiral require reversing *both* the first and second coordinate to be connected to the directed path following the surface of the cube - avoiding the spiral.

Deadlocks

Definition

A connected subset D of $X = I^n \setminus F$ is a deadzone wrt. the order $\vec{I}^{n-k} \times \overset{\leftrightarrow}{I}^{k}$ if for all $d \in D$ and $\gamma : (\vec{I}, 0, 1) \to (X, d, -)$,

- $\gamma(t) \in D$
- $\gamma_j(t)$ is constant for $1 \leq j \leq n - k$

Fix a $d \in D$. Then D is a subset of the hypersurface $x_j = d_j$ $1 \leq j \leq n - k$.
If we consider paths from $\mathbf{0}$ to $\mathbf{1}$, we require $\mathbf{1} \notin D$.

Deadzones - one free coordinate

Let $X = I^n \setminus F$ with partial order $\vec{I}^{n-1} \times \overleftrightarrow{I}$. $F = \cup R^i$, $R^i = [a^i_1, b^i_1] \times \cdots \times [a^i_n, b^i_n]$

Calculating Deadzone

Let $R^{j_1} \cap \cdots \cap R^{j_{n-1}} \cap R^+$ and $R^{j_1} \cap \cdots \cap R^{j_{n-1}} \cap R^-$ be nonempty. Suppose

- $R^{j_1} \cap \cdots \cap R^{j_{n-1}} \cap R^+ = [\tilde{\mathbf{a}}, \tilde{\mathbf{b}}] \times [a^+, b^+]$, where $[\tilde{\mathbf{a}}, \tilde{\mathbf{b}}]$ is the generalized interval, an n-1 dimensional rectangle.
- $R^{j_1} \cap \cdots \cap R^{j_{n-1}} \cap R^- = [\tilde{\mathbf{a}}, \tilde{\mathbf{b}}] \times [a^-, b^-]$

If $b^- < a^+$, then $[\tilde{\mathbf{a}}, \tilde{\mathbf{b}}] \times [b^-, a^+]$ is a deadzone, if it is in $X \setminus F$

The point $(\tilde{\mathbf{a}}, a^+)$ is a deadlock wrt. the order $\vec{I}^n$.

The point $(\tilde{\mathbf{a}}, b^-)$ is a deadlock wrt. the order $\vec{I}^{n-1} \times \overleftarrow{I}$.

Deadzones - general

$D \subset X \setminus F$, $D \subset H = \{x | x_j = d_j, 1 \leq j \leq n-k\}$ is a deadzone wrt. $\vec{I}^{n-k} \times \overset{\leftrightarrow}{I}^k$ if there is an $\varepsilon > 0$ s.t.

- for $x \in D$ and for $j = 1, \ldots, n-k$, $x + te_j \in F$ for $0 < t < \varepsilon$ and e_j the j'th unit vector.
- D is a connected component of $H \setminus F$.

Example

Let $n = 3, k = 2$. Let p^1 be a deadlock wrt. $\vec{I}^3$, p^2 wrt. $\vec{I}^2 \times \overleftarrow{I}$ p^3 wrt. $\vec{I} \times \overleftarrow{I}^2$ and p^4 wrt. $\vec{I} \times \overleftarrow{I} \times \vec{I}$ and suppose they are the corners of an isothetic 2D rectangle D and

- $p^1 = (c, b_1, b_2)$, $p^2 = (c, b_1, a_1)$, $p^3 = (c, a_1, a_2)$, $p^4 = (c, a_1, b_2)$
- The lines connecting p^1, p^2 and p^2, p^3, p^3, p^4 and p^4, p^1 are deadzones wrt to the relevant orders.
- For $x \in D$, $x + t(1, 0, 0) \in F$ for t small enough.
- $D \cap F = \emptyset$

Then D is a deadzone.
This does *not* give all the deadzones:
Deadzones need not be k-rectangles with deadlocks in the corners. The boundary could be more complicated. Think of a 2D “wedge” for $n = 3, k = 2$

Unsafe area wrt. a deadzone

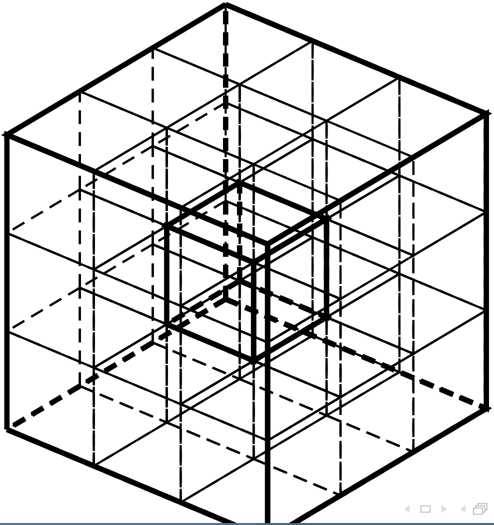
Definition

Let $D \in X \setminus F$ be a deadzone. A point $x \in X \setminus F$ is in the unsafe area wrt. D, if all dipaths initiating in x will end in D.

We do not allow paths to stop for no reason, or to stay in a hyperplane $(y_1, \ldots, y_{n-k}) \times \overset{\leftrightarrow}{I}^k$.
The immediate unsafe area is bounded by the rectangles bounding D.
Need general calculation. In simple examples, this is the intersection of the unsafe areas wrt. the deadlocks cornering D.

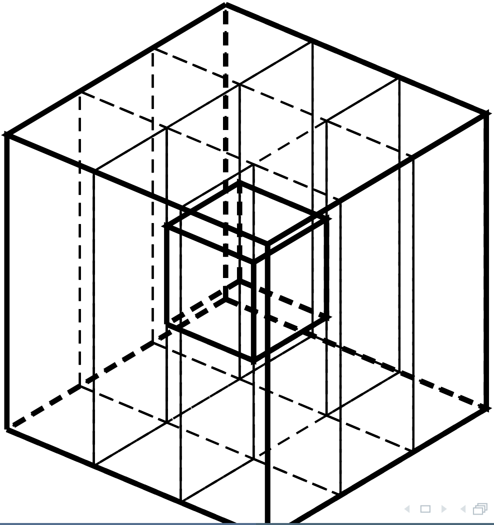
Components

Components are derived from an lr-system by formally inverting "inessentials".



Components

Components are derived from an lr-system by formally inverting "inessentials".



Conclusions and Future work

- We compare the geometric models with and without reversibility
- The lattice of d-structures provides an iterative strategy for reversing processes, inserting cubes,...
- Want calculations of trace spaces (or perhaps homology directly)
- Want calculations of relative homology and homotopy
- Want comparison with process algebra approach - RCSS